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On the Dynamics of Geometric Quadratic Stochastic Operator Generated by 2-Partition on Countable State Space

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Abstract

A quadratic stochastic operator (QSO) is frequently acknowledged as the analysis source to investigate dynamical properties and modeling in numerous areas. Countless classes of QSO have been investigated since the operator was introduced in 1920s. The study of QSOs is still an open problem in the nonlinear operator theory field, especially QSOs on infinite state space. We are interested in the dynamics of Geometric QSO generated by a 2-partition defined on countable state space. We first show the system of equations formed from defined Geometric QSO with infinite-dimensional space can be simplified into a one-dimensional setting, corresponding to the number of defined partitions. The trajectory behavior of such a system is investigated by using functional analysis approach, where the operator either converges to a unique fixed point or has a second-order cycle. It is shown that such an operator can be either regular or nonregular for arbitrary initial points depending on the value of parameters. In this research, we present two cases, i.e., two different parameters and three different parameters. We display the form of the fixed point and periodic points of period-2. Moreover, an example for the nonregular transformation will be provided, where such QSO has 2-periodic points.

Keywords: geometric distribution; quadratic stochastic operator; measurable partition; countable set; regularity.

1 Introduction

The theory of quadratic stochastic operators (QSOs) was first introduced in the early 20th century by Bernstein [1]. Since then, QSOs have been frequently applied to understand and describe dynamical properties and mathematical modeling in diverse fields.

Let us recall some notions, notations, and definitions related to QSOs. Let $(X, \mathcal{F})$ be a measurable space, where X is a state space and $\mathcal{F}$ is a σ-algebra, then $S(X, \mathcal{F})$ is the set of all probability measures on $(X, \mathcal{F})$.

If a state space $X = \{1, 2, \ldots, m\}$ is finite and the corresponding σ-algebra is the power set $P(X)$, then the set of all probability measures on $(X, P(X))$ is called $(m-1)$-dimensional simplex with the following form

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \geq 0,\ i = 1, ..., m,\ \sum_{i=1}^{m} x_i = 1 \right\}.$$

A mapping $V : S^{m-1} \to S^{m-1}$ is known as a quadratic stochastic operator (QSO) with

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \tag{1}$$

where

$$P_{ij,k} \geq 0,\ P_{ij,k} = P_{ji,k},\ \sum_{k=1}^{m} P_{ij,k} = 1\ \ for\ \ i, j, k \in X, \tag{2}$$

for $i, j, k \in X$. For a given $\mathbf{x}^{(0)} \in S^{(m-1)}$, the trajectory $\mathbf{x}^{(n)}$ of $\mathbf{x}^{(0)}$ under the action of QSO in (1) is defined by $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \ldots$. In other words, for any initial point $\lambda \in S(X, \mathcal{F})$, where $V^{n+1}\lambda = V(V^n\lambda)$, then $\{V^n\lambda : n = 0, 1, 2, ...\}$ is the trajectory of such initial point λ.

Definition 1.1. *A quadratic stochastic operator V is called a regular if for any initial point $\lambda \in S(X, \mathcal{F})$ the limit*

$$\lim_{n\to\infty} V^n(\lambda)$$

exists.

The main problem in the nonlinear operator theory is centered on the study of the asymptotic behavior of these trajectories which is still not fully solved even in the simplest nonlinear operator form, i.e., QSO. Over almost a century after being presented by Bernstein as a mathematical solution to the problem of heredity, QSOs have been widely studied, where different classes of QSO are introduced and further investigated either on finite or infinite state space in many publications. Here we shall consider the QSO defined on infinite state space.

The classes of QSO defined on infinite state space have been originally inspired by the study of infinite-dimensional quadratic Volterra operators by Mukhamedov in [7]. Thus, the studies of different classes of QSO on infinite state space specifically in countable state space have grown constantly throughout the 21st century. It can be seen from [2] and [5] in which the authors introduced Poisson, Geometric and mixing both distributions as the probability measure in the QSO on countable state space.

Further research on Geometric QSOs defined on countable sets are presented in [3] and [4]. The authors constructed the Geometric QSO generated by a 2-partition of singleton and infinite points, respectively, and investigated the trajectory behavior of such an operator with two different parameters in terms of its regularity. The study showed that such QSO is regular. Generally, a Geometric QSO generated by a 2-partition on countable state space can be simplified as a QSO in a one-dimensional simplex studied by Lyubich in [6]. Other than the mentioned QSOs, many different classes of QSO defined on lower-dimensional simplex have been studied in numerous publications such as b-bistochastic quadratic stochastic operators, (see [8] and [9]).

Motivated by the study of Geometric QSO in [3] and [4], in this research, we shall study the dynamics of Geometric QSO generated by 2-partition with two and three different parameters. Later, we shall discuss the form and type of fixed point for the regular transformation as well as the form of periodic point for the nonregular case. We also shall present an example to describe the form of the periodic point of the nonregular transformation for some fixed parameters.

2 Geometric Quadratic Stochastic Operator Generated by 2-Partition

As mentioned previously, we consider nonlinear transformations defined on countable state space and investigate their dynamics. Let $X = 0, 1, 2, \ldots$ be a countable sample space and corresponding σ-algebra $\mathcal{F}$ be the power set $P(X)$. A probability measure μ on countable sample space X is defined as measure $\mu(k)$ written as $\mu(k)$ of each singleton $\{k\}$, $k \in X$. For countable state space X, a QSO V has the following form:

$$V\mu\left(k\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu\left(i\right)\mu\left(j\right), \tag{3}$$

where $\mu \in S\left(X, \mathcal{F}\right)$ and $P_{ij,k}$ satisfies the following conditions:

$$P_{ij,k} \geq 0,\;\; P_{ij,k} = P_{ji,k},\;\; \sum_{k=1}^{\infty} P_{ij,k} = 1 \;\; for \;\; i,j,k \in X.$$

Note that a Geometric distribution, G_r with a real parameter r, $0 < r < 1$, is defined on the countable set X as

$$G_r\left(k\right) = \left(1-r\right)r^k,\;\; k \in X.$$

Definition 2.1. *A QSO V (3) is called a Geometric QSO, if for any $i, j \in X$, the probability measure $P(i, j, \cdot)$ is the Geometric distribution with parameter $r(i, j)$, where $0 < r(i, j) < 1$.*

One can define a Geometric QSO generated by 2-partition as follows. Let $\xi = \{A_1, A_2\}$ be a measurable 2-partition of the set X and $\zeta = \{B_{11}, B_{22}, B_{12}\}$ be a corresponding partition of the Cartesian product $X \times X$, where $B_{pp} = A_p \times A_p$ for $p = 1, 2$ and $B_{12} = (A_1 \times A_2) \cup (A_2 \times A_1)$. We select a family $\left\{\mu_{pq}^G : p,q=1,2\right\}$ of Geometric distributions with parameters $r_{11} = r_1$, $r_{22} = r_2$, $r_{12} = r_3$, and if $(i,j) \in B_{pq}$, we define probability measure $P(i,j,A)$ as follows:

$$P\left(i,j,A\right) = \mu_{pq}^G\left(A\right) \text{ if } \left(i,j\right) \in B_{pq},\ p,q = 1,2, \tag{4}$$

for $A \in \mathcal{F}$. Then for arbitrary $\lambda \in S(X,\mathcal{F})$,

$$\begin{aligned}
\left(V\lambda\right)\left(A\right) &= \int\limits_X \int\limits_X P\left(i,j,A\right) d\lambda\left(i\right) d\lambda\left(j\right) \\
&= \sum_{p,q=1}^{2} \int\limits_{A_p} \int\limits_{A_q} \mu_{pq}^G\left(A\right) \cdot d\lambda\left(i\right) d\lambda\left(j\right) \\
&= \sum_{p,q=1}^{2} \mu_{pq}^G\left(A\right) \lambda\left(A_p\right) \lambda\left(A_q\right).
\end{aligned}$$

By using mathematical induction, we will get

$$\begin{aligned}
\left(V^{n+1}\lambda\right)\left(A\right) &= \int\limits_X \int\limits_X P\left(i,j,A\right) dV^n\lambda\left(i\right) dV^n\lambda\left(j\right) \\
&= \sum_{p,q=1}^{2} \int\limits_{A_p} \int\limits_{A_q} \mu_{pq}^G\left(A\right) \cdot dV^n\lambda\left(i\right) dV^n\lambda\left(j\right) \\
&= \sum_{p,q=1}^{2} \mu_{pq}^G\left(A\right) \left(V^n\lambda\right)\left(A_p\right) \left(V^n\lambda\right)\left(A_q\right),
\end{aligned}$$

with

$$\left(V^{n+1}\lambda\right)\left(A_k\right) = \sum_{p,q=1}^{2} \mu_{pq}^G\left(A_k\right) \left(V^n\lambda\right)\left(A_p\right) \left(V^n\lambda\right)\left(A_q\right), \tag{5}$$

for $k = 1,2$.

Assume $x_k{}^{(n)} = \left(V^n\lambda\right)\left(A_k\right)$ and $P_{ij,k} = \mu_{pq}^G\left(A_k\right)$ for $(i,j) \in B_{pq}$. Then $\left(x_1{}^{(n)}, x_2{}^{(n)}, ..., x_m{}^{(n)}\right) \in S^{m-1}$ and the system of equations in (5) can be rewritten as follows:

$$\left(W\mathbf{x}\right)_k = \sum_{p,q=1}^{2} P_{ij,k} x_p x_q, \tag{6}$$

for $k = 1,2$.

A quadratic stochastic operator W on S^1 has the following form

$$W : \begin{cases} x_1{}' = a_{11}x_1{}^2 + 2a_{12}x_1x_2 + a_{22}x_2{}^2, \\ x_2{}' = b_{11}x_1{}^2 + 2b_{12}x_1x_2 + b_{22}x_2{}^2, \end{cases} \tag{7}$$

where $a_{11} = \mu_{11}^G\left(A_1\right),\ a_{12} = \mu_{12}^G\left(A_1\right),\ a_{22} = \mu_{22}^G\left(A_1\right), b_{11} = \mu_{11}^G\left(A_2\right),\ b_{12} = \mu_{12}^G\left(A_2\right),$ and $b_{22} = \mu_{22}^G\left(A_2\right)$ are arbitrary coefficients with $0 < a_{pq}, b_{pq} < 1$ for $p,q = 1,2$, where $a_{pq} + b_{pq} = 1$.

2.1 Dynamics of Geometric Quadratic Stochastic Operator Generated by 2-Partition with Two Parameters

The system of equations (7) have been studied in [6]. Assume

$$\Delta = 4\left(1-a_{11}\right)a_{22} + \left(1-2a_{12}\right)^2, \lambda = 1-\sqrt{\Delta}.$$

Lemma 2.1. *For the system of equations (7), if $r_1 = r_2 \neq r_3$, then $0 < \Delta < 2$ and $|\lambda| < 1$. Hence, x_1^* is an attracting fixed point.*

Proof. If $r_1 = r_2 \neq r_3$, then we have $a_{11} = a_{22} \neq a_{12}$ and below is the corresponding system of equations.

$$W : \begin{cases} {x_1}' = a_{11}({x_1}^2 + {x_2}^2) + 2a_{12}x_1x_2, \\ {x_2}' = (1-a_{11})({x_1}^2 + {x_2}^2) + 2(1-a_{12})x_1x_2. \end{cases} \tag{8}$$

Since $x_2 = 1 - x_1$, it follows that ${x_2}' = 1 - {x_1}'$. By substituting $x_2 = 1 - x_1$ into the first equation in (8), we may have a quadratic equation with respect to x_1. Assume that ${x_1}' = x_1$, we may solve the following quadratic equation:

$$x_1 = 2\left(a_{11}-a_{12}\right)x_1^2 - 2\left(a_{11}-a_{12}\right)x_1 + a_{11}. \tag{9}$$

By solving the quadratic equation in (9), one can find that the fixed point is unique and belongs to the interval $(0, 1)$ with the following form:

$$x_1^* = \frac{2\left(a_{11}-a_{12}\right) + 1 - \sqrt{\Delta}}{4\left(a_{11}-a_{12}\right)}.$$

Note that Δ is the discriminant of the quadratic equation in (9), where $a_{11} = a_{22}$. It is easy to observe that one will get $0 < \Delta < 2$ since $a_{11}, a_{12} \in (0, 1)$.

Remark 2.1. *Let x^* be a fixed point and $f'\left(x\right)$ be the first derivative of a function $f\left(x\right)$. Then, the following statements hold true:*

1. *if $\left|f'\left(x^*\right)\right| < 1$, then x^* is an attracting fixed point,*
2. *if $\left|f'\left(x^*\right)\right| > 1$, then x^* is a repelling fixed point.*

After simple algebra, we have

$$f'\left(x_1\right) = 4\left(a_{11}-a_{12}\right)x_1 - 2\left(a_{11}-a_{12}\right),$$

such that $\lambda = f'\left(x_1^*\right)$ and $1-\sqrt{2} < \lambda < 1$. Due to Remark 2.1, one can conclude that the quadratic equation in (9) has an attracting fixed point, (i.e. stable unique fixed point). This completes the proof.

□

2.2 Dynamics of Geometric Quadratic Stochastic Operator Generated by 2-Partition with Three Parameters

Lemma 2.2. *For the system of equations (7), if $r_1 \neq r_2 \neq r_3$, then $0 < \Delta < 5$ and $|\lambda| < 1$ when $0 < \Delta < 4$, and $|\lambda| > 1$ when $4 < \Delta < 5$. Hence, x_1^* is a repelling fixed point.*

Proof. If $r_1 \neq r_2 \neq r_3$, then we have $a_{11} \neq a_{22} \neq a_{12}$. Referring to the system of equations (7) and assuming $x_1{}' = x_1$, one may have a quadratic equation with respect to x_1 as follows:

$$x_1 = (a_{11} - 2a_{12} + a_{22})\, x_1^2 + 2\,(a_{12} - a_{22})\, x_1 + a_{22}. \tag{10}$$

According to Lyubich in [6], the quadratic equation (10) has a unique fixed point that belongs to the open interval $(0, 1)$ in the following form:

$$x_1^* = \frac{-2\,(a_{12} - a_{22}) + 1 - \sqrt{\Delta}}{2\,(a_{11} - 2a_{12} + a_{22})},$$

where Δ is the discriminant of the quadratic equation in (10). Given the fact that $0 < a_{ij} < 1$ for $i, j = 1, 2$, it is evident that $0 < \Delta < 5$.

Let

$$f'\,(x_1) = 2\,(a_{11} - 2a_{12} + a_{22})\, x_1 + 2\,(a_{12} - a_{22})\,,$$

be the first derivative of the right-hand side of equation (10), then we have $\lambda = f'\,(x_1^*)$. One can easily verify that if $0 < \Delta < 4$, then $\lambda < 1$ and x_1^* is an attracting fixed point. Meanwhile, if $4 < \Delta < 5$, then $\lambda > 1$ and x_1^* is a repelling fixed point. Thus, the proof is complete.

□

2.3 Form of Point of Nonregular Geometric Quadratic Stochastic Operator with Three Parameters

Based on the above lemmas, one can say that only the Geometric QSO generated by 2-partition with three different parameters may have a repelling fixed point. Recall that a repelling fixed point indicates that all trajectories of the operator in (7) tend to a cycle of second-order except when $x_1^{(0)} = x_1^*$. Hence, we can find the periodic points of period 2 of the right-hand side of the equation in (10) by considering the following function:

$$\varphi^2\,(x_1) = \varphi\,(\varphi\,(x_1)) = x_1, \tag{11}$$

where $\varphi\,(x_1)$ is the function on the right-hand side of the equation in (10).

Let

$$\varphi\,(x_1) = Ax_1{}^2 + Bx_1 + C, \tag{12}$$

where $A = a_{11} - 2a_{12} + a_{22}$, $B = 2\,(a_{12} - a_{22})$, and $C = a_{22}$. Then,

$$\begin{aligned}\varphi^2\,(x_1) &= A\left(Ax_1{}^2 + Bx_1 + C\right)^2 + B\left(Ax_1{}^2 + Bx_1 + C\right) + C\\ &= A^3x_1{}^4 + 2A^2Bx_1{}^3 + \left(2A^2C + AB^2 + AB\right)x_1{}^2 + \left(2ABC + B^2\right)x_1\\ &+ AC^2 + BC + C.\end{aligned}$$

To solve the equation in (11), one may get

$$\left(A^2x_1{}^2 + (AB + A)x_1 + AC + B + 1\right)\left(Ax_1{}^2 + Bx_1 + C - x_1\right) = 0. \tag{13}$$

Consequently, one can find the roots for the following quadratic equations:

$$A^2{x_1}^2 + (AB + A)x_1 + AC + B + 1 = 0, \tag{14}$$

$$A{x_1}^2 + Bx_1 + C - x_1 = 0. \tag{15}$$

After simple algebra, the quadratic equation in (14) has the following roots:

$$x_{1,a}^* = \frac{-(B+1) + \sqrt{B^2 - 2B - 3 - 4AC}}{2A},$$
$$x_{1,b}^* = \frac{-(B+1) - \sqrt{B^2 - 2B - 3 - 4AC}}{2A}.$$

Meanwhile, the roots for the quadratic equation in (15) are as follows:

$$x_{1,c}^* = \frac{-(B-1) + \sqrt{(B-1)^2 - 4AC}}{2A},$$
$$x_{1,d}^* = \frac{-(B-1) - \sqrt{(B-1)^2 - 4AC}}{2A}.$$

One may observe that $x_{1,c}^*$ and $x_{1,d}^*$ are the fixed points of the quadratic equation in (10), where $x_{1,c}^* \notin (0,1)$, $x_{1,d}^* \in (0,1)$, and $x_{1,d}^* = x_1^*$. Thus, $x_{1,a}^*$ and $x_{1,b}^*$ are the periodic points of period 2 of the system of equations in (7).

Lemma 2.3. *If $4 < \Delta < 5$, then the operator (7) has 2-periodic points, $(x_{1,a}^*, x_{2,a}^*)$ and $\left(x_{1,b}^*, x_{2,b}^*\right)$, which are different from the fixed point (x_1^*, x_2^*).*

Proof. It can be seen from $|\varphi'(x_1{}^*)| = \left|1 - \sqrt{\Delta}\right|$ that if $4 < \Delta < 5$, then x_1^* is a repelling fixed point of $\varphi(x_1)$ and the function $\varphi^2(x_1)$ has the fixed points x_1^*, $x_{1,a}^*$, and $x_{1,b}^*$ that belong to the interval $(0,1)$, i.e., $\varphi^2\left(x_{1,a}^*\right) = x_{1,a}^*$, and $\varphi^2\left(x_{1,b}^*\right) = x_{1,b}^*$. Since $x_2 = 1 - x_1$, it is easy to obtain x_2^*, $x_{2,a}^*$, and $x_{2,b}^*$. This completes the proof. □

3 Regularity of Geometric Quadratic Stochastic Operator Generated by 2-Partition

Based on Definition 1.1, the regularity of a QSO is determined by the existence of the limit of such an operator. In this section, we provide theorems to describe the trajectory behavior of the Geometric QSO generated by 2-partition with two and three different parameters in terms of its regularity.

Theorem 3.1. [6] *If $0 < \Delta < 4$, then a one-dimensional QSO (7) is a regular, and if $4 < \Delta < 5$, then there exists a cycle of second-order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.*

Let $\left(\varphi^2\right)'(x_1)$ be the first derivative of the function $\varphi^2(x_1)$, where

$$\left(\varphi^2\right)'(x_1) = 2A\left(Ax_1^2 + Bx_1 + C\right)(2Ax_1 + B) + B(2Ax_1 + B). \tag{16}$$

By substituting $x_{1,a}^*$ and $x_{1,b}^*$ into the function in (16), it will yield

$$\left(\varphi^2\right)'\left(x_{1,a}^*\right) = \left(\varphi^2\right)'\left(x_{1,b}^*\right) = 4a_{22}(a_{11} - 1) - 4a_{12}(a_{12} - 1) + 4 = 5 - \Delta.$$

Since we have $4 < \Delta < 5$, then we may obtain $|5 - \Delta| < 1$. This leads to the fact that $\left|\left(\varphi^2\right)'\left(x_{1,a}^*\right)\right| < 1$ and $\left|\left(\varphi^2\right)'\left(x_{1,b}^*\right)\right| < 1$. Hence, this shows that the fixed points $x_{1,a}^*$ and $x_{1,b}^*$ of the function $\varphi^2(x_1)$ are attracting.

Theorem 3.2. *Let* $0 < \Delta < 5$.

1. *If* $0 < \Delta < 4$, *then there exists an open set* $\mathcal{U} \subset S^1$ *such that* $\mathbf{x}^* \in \mathcal{U}$ *and for any* $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) \in \mathcal{U}$ *we have*

$$\lim_{n\to\infty} W^n\left(x_1^{(0)}, x_2^{(0)}\right) = \left(x_1^*, x_2^*\right).$$

It follows that such an operator W *is a regular transformation.*

2. *If* $4 < \Delta < 5$ *and* $a_{12} < \frac{1}{2} - \sqrt{1 - (1 - a_{11})a_{22}}$, *then there exists an open set* $\mathcal{U} \subset S^1$ *such that* $\mathbf{x}_a^*, \mathbf{x}_b^* \in \mathcal{U}$ *and for any* $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) \in \mathcal{U}$ *we have*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{x}_a^*, & \text{if } x_1^{(0)} < x_1^* \text{ and } n = 2k+1 \text{ or } x_1^{(0)} > x_1^* \text{ and } n = 2k, \\ \mathbf{x}^*, & \text{if } x_1^{(0)} = x_1^*, \\ \mathbf{x}_b^*, & \text{if } x_1^{(0)} < x_1^* \text{ and } n = 2k \text{ or } x_1^{(0)} > x_1^* \text{ and } n = 2k+1. \end{cases}$$

3. *If* $4 < \Delta < 5$ *and* $a_{12} > \frac{1}{2} + \sqrt{1 - (1 - a_{11})a_{22}}$, *then there exists an open set* $\mathcal{U} \subset S^1$ *such that* $\mathbf{x}_a^*, \mathbf{x}_b^* \in \mathcal{U}$ *and for any* $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) \in \mathcal{U}$ *we have*

$$\lim_{n\to\infty} W^n\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{x}_a^*, & \text{if } x_1^{(0)} < x_1^* \text{ and } n = 2k \text{ or } x_1^{(0)} > x_1^* \text{ and } n = 2k+1, \\ \mathbf{x}^*, & \text{if } x_1^{(0)} = x_1^*, \\ \mathbf{x}_b^*, & \text{if } x_1^{(0)} < x_1^* \text{ and } n = 2k+1 \text{ or } x_1^{(0)} > x_1^* \text{ and } n = 2k, \end{cases}$$

where $\mathbf{x}^* = \left(x_1^*, x_2^*\right)$ *is fixed point and* $\mathbf{x}_a^* = \left(x_{1,a}^*, x_{2,a}^*\right)$, $\mathbf{x}_b^* = \left(x_{1,b}^*, x_{2,b}^*\right)$ *are periodic points described above. It follows that such an operator* W *is a nonregular transformation.*

Proof. 1) For $0 < \Delta < 4$, it can be seen from $\varphi'\left(x_1^*\right) = 1 - \sqrt{\Delta}$ that $x_1^* = \frac{-2(a_{12}-a_{22})+1-\sqrt{\Delta}}{2(a_{11}-2a_{12}+a_{22})} \in (0, 1)$ is an attracting hyperbolic fixed point. Therefore, $x_1^{(n)}$ will converge to $x_1^{(*)}$ when $0 < \Delta < 4$. A trajectory of the operator (7) on the invariant set $\gamma = \left\{(x_1, x_2) \in S^1 : x_1 = x_1^*\right\}$ is as follows:

$$\mathbf{x}^{(n)} = \left(x_1^{(n)}, 1 - x_1^{(n)}\right),$$

where $x_1^{(n)}$ satisfies the equality

$$x_1^{(n+1)} = (a_{11} - 2a_{12} + a_{22})\left(x_1^*\right)^2 + 2(a_{12} - a_{22})x_1^* + a_{22}. \tag{17}$$

It follows from (17) that $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Therefore, $\lim_{n\to\infty} W^n\left(x_1^{(0)}, x_2^{(0)}\right) = (x_1^*, x_2^*)$ and the operator W is regular.

2) Next, when $4 < \Delta < 5$, it is easy to observe from $|\varphi'(x_1{}^*)| = \left|1-\sqrt{\Delta}\right|$ that $x_1{}^*$ is a repelling hyperbolic fixed point of $\varphi(x_1)$. Additionally, when $4 < \Delta < 5$, the fixed points $x_{1,a}^*$ and $x_{1,b}^*$ of function $\varphi^2(x_1)$ are attracting, which follows from $\left|(\varphi^2)'(x_{1,a}^*)\right| < 1$ and $\left|(\varphi^2)'\left(x_{1,b}^*\right)\right| < 1$.

For the case where $4 < \Delta < 5$, we shall consider all possible cases on b. We use the fact that we will obtain $4 < \Delta < 5$, if and only if $a_{11} \in \left[0, \frac{1}{4}\right]$, $a_{22} > \frac{3}{4(1-a_{11})}$, and $a_{12} < \frac{1}{2} - \sqrt{1-(1-a_{11})\,a_{22}}$ or $a_{12} > \frac{1}{2} + \sqrt{1-(1-a_{11})\,a_{22}}$. Now we show the trajectory behavior of the operator W in (7) when $4 < \Delta < 5$ and $a_{12} < \frac{1}{2} - \sqrt{1-(1-a_{11})\,a_{22}}$.

Define the operator $W : [0,1] \to [0,1]$:

$$W : \begin{cases} x'_1 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \\ x'_2 = (1-a_{11})\,x_1^2 + 2\,(1-a_{12})\,x_1x_2 + (1-a_{22})\,x_2^2. \end{cases} \tag{18}$$

The operator W has the following invariant sets:

$$\gamma = \left\{(x_1, x_2) \in S^1 : x_1 = x_1^*\right\}$$

and

$$\Gamma = \left\{(x_1, x_2) \in S^1 : x_1 = x_{1,a}^* \text{ or } x_1 = x_{1,b}^*\right\}.$$

Note that if $\mathbf{x}^{(0)} \in \gamma$, then $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. If $\mathbf{x}^{(0)} \in \Gamma$, then we will get $\lim_{n\to\infty} x_1^{(2n)} = x_{1,a}^*$ and $\lim_{n\to\infty} x_1^{(2n+1)} = x_{1,b}^*$ when $x_1^{(0)} = x_{1,a}^*$. Consequently, we have $\lim_{n\to\infty} x_1^{(2n)} = x_{1,b}^*$ and $\lim_{n\to\infty} x_1^{(2n+1)} = x_{1,a}^*$ when $x_1^{(0)} = x_{1,b}^*$ for $n = 0, 1, 2, \ldots$. This is due to the fact that $x_{1,a}^*$ and $x_{1,b}^*$ are the attracting fixed points of the function $f^2(x_1)$.

Now we shall consider $\mathbf{x}^{(0)} \in S^1 \backslash \gamma$. This leads us to two cases, where $x_1^{(0)} < x_1^*$ and $x_1^{(0)} > x_1^*$. One may observe that $x_{1,b}^* < x_1^* < x_{1,a}^*$ when $a_{12} < \frac{1}{2} - \sqrt{1-(1-a_{11})\,a_{22}}$. Hence, if $x_1^{(0)} < x_1^*$, we will have $\varphi\left(x_1^{(0)}\right) > x_1^*$ and $\varphi^2\left(x_1^{(0)}\right) < x_1^*$. Since $x_{1,a}^*$ and $x_{1,b}^*$ are attracting fixed points of the function $\varphi^2(x_1)$, then we may conclude that $\lim_{n\to\infty} \varphi^{2n}\left(x_1^{(0)}\right) = x_{1,b}^*$ and $\lim_{n\to\infty} \varphi^{2n+1}\left(x_1^{(0)}\right) = x_{1,a}^*$ for $n = 0, 1, 2, \ldots$.

Similarly, if $x_1^{(0)} < x_1^*$, one will obtain $\varphi\left(x_1^{(0)}\right) < x_1^*$ and $\varphi^2\left(x_1^{(0)}\right) > x_1^*$. This gives us $\lim_{n\to\infty} \varphi^{2n}\left(x_1^{(0)}\right) = x_{1,a}^*$ and $\lim_{n\to\infty} \varphi^{2n+1}\left(x_1^{(0)}\right) = x_{1,b}^*$ for $n = 0, 1, 2, \ldots$.

3) As we have established the previous proof, then it is easy to show the trajectory behavior of the operator W when $4 < \Delta < 5$ and $a_{12} > \frac{1}{2} + \sqrt{1-(1-a_{11})\,a_{22}}$. Recall that if $\mathbf{x}^{(0)} \in \gamma$, then $\lim_{n\to\infty} x_1^{(n)} = x_1^*$. Next, when $a_{12} > \frac{1}{2} + \sqrt{1-(1-a_{11})\,a_{22}}$, we have $x_{1,a}^* < x_1^* < x_{1,b}^*$. By considering the case when $x_1^{(0)} < x_1^*$, one can conclude that $\lim_{n\to\infty} \varphi^{2n}\left(x_1^{(0)}\right) = x_{1,a}^*$ and $\lim_{n\to\infty} \varphi^{2n+1}\left(x_1^{(0)}\right) = x_{1,b}^*$ for $n = 0, 1, 2, \ldots$ as $\varphi\left(x_1^{(0)}\right) > x_1^*$ and $\varphi^2\left(x_1^{(0)}\right) < x_1^*$.

We handle the case, where $x_1^{(0)} > x_1^*$ similarly. Substituting $x_1^{(0)} > x_1^*$ into the function φ, one may obtain $\varphi\left(x_1^{(0)}\right) < x_1^*$ and $\varphi^2\left(x_1^{(0)}\right) > x_1^*$. Therefore, $\lim_{n\to\infty} \varphi^{2n}\left(x_1^{(0)}\right) = x_{1,b}^*$ and $\lim_{n\to\infty} \varphi^{2n+1}\left(x_1^{(0)}\right) = x_{1,a}^*$ for $n = 0, 1, 2, \ldots$.

It is shown that the operator W in (7) is nonregular when $4 < \Delta < 5$. The proof is complete.

□

Next, we provide an example for a QSO in (7) having periodic points of period 2 in a one-dimensional simplex.

Example 3.1. *Let $A_1 = \{0\}$ and $A_2 = \{1, 2, ...\}$ be a measurable 2-partition for the Geometric QSO generated by 2-partition. For any $\frac{7}{8} < \varepsilon < 1$, we define $r_1 = \varepsilon$, $r_2 = \frac{1}{20}\varepsilon$, and $r_3 = \frac{1}{10}\varepsilon$. Then, we will have*

$$\varphi(x_1) = (-\varepsilon)\, x_1^2 + \left(\frac{1}{10}\varepsilon\right) x_1 + \left(1 - \frac{1}{10}\varepsilon\right),$$

where $\Delta_\varepsilon = -\frac{39}{100}\varepsilon^2 + \frac{19}{5}\varepsilon + 1$. One can easily observe that $4 < \Delta_\varepsilon < 5$. Then, the above function will have a fixed point, $x_1^* = \frac{10\sqrt{\Delta_\varepsilon}+\varepsilon-10}{20\varepsilon} \in (0, 1)$. Next, we shall search for the periodic points of period 2 by considering the function $\varphi^2(x_1)$. By solving the quartic equation as in (11), one will get the fixed points of the function $\varphi^2(x_1)$ that belong to the interval $(0, 1)$ with the following form:

$$x_1^* = \frac{10\sqrt{\Delta_\varepsilon} + \varepsilon - 10}{20\varepsilon},$$
$$x_{1,a}^* = \frac{\varepsilon + 10 + 10\sqrt{\Delta_\varepsilon - 4}}{20\varepsilon},$$
$$x_{1,b}^* = \frac{\varepsilon + 10 - 10\sqrt{\Delta_\varepsilon - 4}}{20\varepsilon}.$$

Hence, in this case, we obtain that $Fix(W_\varepsilon) = \left\{\left(\frac{10\sqrt{\Delta_\varepsilon}+\varepsilon-10}{20\varepsilon}, \frac{10\sqrt{\Delta_\varepsilon}+19\varepsilon-10}{20\varepsilon}\right)\right\}$ and $Per_2(W_\varepsilon) = \left\{\left(\frac{\varepsilon+10+10\sqrt{\Delta_\varepsilon-4}}{20\varepsilon}, \frac{19\varepsilon+10+10\sqrt{\Delta_\varepsilon-4}}{20\varepsilon}\right), \left(\frac{\varepsilon+10-10\sqrt{\Delta_\varepsilon-4}}{20\varepsilon}, \frac{19\varepsilon+10-10\sqrt{\Delta_\varepsilon-4}}{20\varepsilon}\right)\right\}$ for any $\frac{7}{8} < \varepsilon < 1$.

□

Given that such an operator W in (7) has a unique fixed point, i.e., $|Fix(W)| = 1$ for any initial point $\mathbf{x}^{(0)}$, where $0 < \Delta < 4$, one can conclude that the trajectory behavior of the operator converges to a stable fixed point. This demonstrates that such an operator is regular. Meanwhile, the existence of the repelling fixed points when $4 < \Delta < 5$ for a quadratic equation in (10) suggests that there exists a second-order cycle for the operator in (7). Based on Definition 1.1, if the strong limit of such an operator does not exist, then it is not regular.

4 Conclusions

As mentioned in the introduction, the quadratic stochastic operator (QSO) generated by a 2-measurable partition defined on countable state space can be reduced into a one-dimensional QSO.

This paper presented the dynamics of the Geometric QSO generated by a 2-measurable partition by analyzing the formed quadratic equation (10) while considering two cases, i.e., (i) two different parameters, where $r_1 = r_2 \neq r_3$ and (ii) three different parameters, where $r_1 \neq r_2 \neq r_3$. It is proven that such an operator can be either regular demonstrated by the existence of an attracting unique fixed point or nonregular when there exists a repelling fixed point, where such an operator shows a cycle of second-order. The form of the fixed point and the 2-periodic points can be found by solving the quartic equation in (12).

Observing the obtained result makes it possible to study the QSO on countable state space with more partitions in future research. A complete understanding of such QSOs on infinite state space would significantly contribute to mathematical genetics and dynamical systems.

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